Grade 6 Math Circles

October 3/4/5, 2023

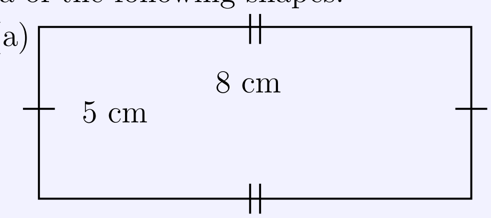
Area, Volume, and Optimization - Solutions

Exercise Solutions

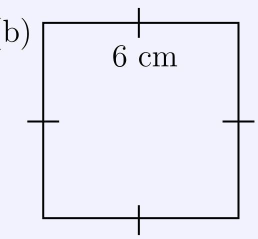
Exercise 1

Calculate the area of the following shapes:

(a)



(b)



Exercise 1 Solution

(a) $A = lw$

$A = (8)(5)$

$A = 40 \text{ cm}^2$

(b) $A = lw$ or $A = s^2$

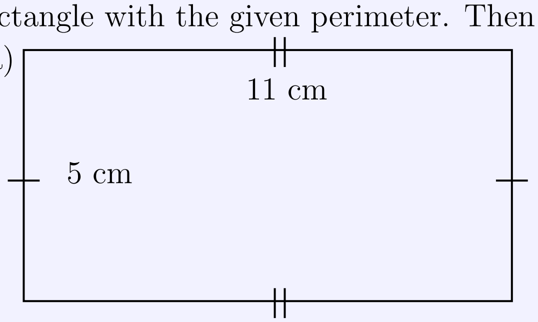
$A = 6 \times 6$ or $(6)^2$

$A = 36 \text{ cm}^2$

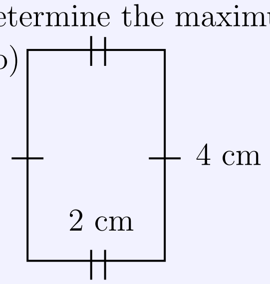
Exercise 2

For each of the following rectangles, determine the dimensions that optimize the area of the rectangle with the given perimeter. Then determine the maximum area.

(a)



(b)



(c) A rectangle with a perimeter of 52 m

Exercise 2 Solution

The optimal shape is a square. In each part, let s be the length of the side of the square.

(a) $P = 2(11 + 5)$

$P = 32$ cm

A square has four equal sides, so each side is 8 cm.

$A = 8 \times 8 \text{ cm}^2$

$A = 64 \text{ cm}^2$

(b) $P = 2(2 + 4)$

$P = 12$ cm

A square has four equal sides, so each side is 3 cm.

$A = 3 \times 3 \text{ cm}^2$

$A = 9 \text{ cm}^2$

(c) $P = 52$

A square has four equal sides, so each side is 13 cm.

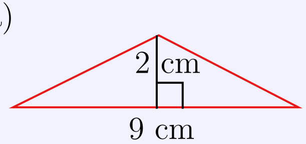
$A = 13 \times 13 \text{ cm}^2$

$A = 169 \text{ cm}^2$

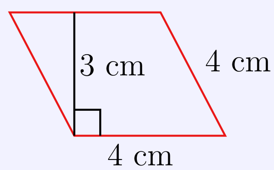
Exercise 3

Calculate the area of the following shapes.

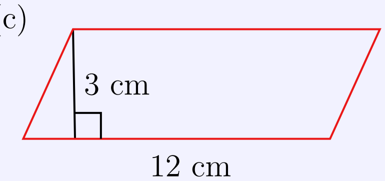
(a)



(b)



(c)



Exercise 3 Solution

(a) $A = \dfrac{(9)(2)}{2}$

$A = 9 \text{ cm}^2$

(b) $A = (4)(3)$

$A = 12 \text{ cm}^2$

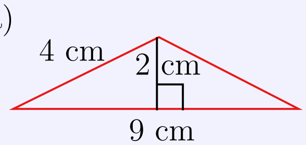
(c) $A = (12)(3)$

$A = 36 \text{ cm}^2$

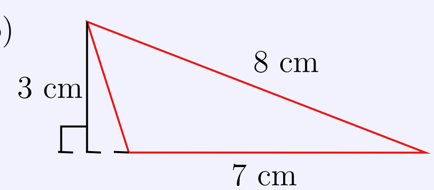
Exercise 4

First determine the current area of each triangle. Then determine the maximum area of each triangle with two fixed side lengths.

(a)



(b)



Exercise 4 Solution

The optimal shape is a right-angled triangle. This means the base and height are equal to the two given side lengths.

(a) Current Area:

$A = (9)(2) \div 2 \text{ cm}^2$

$A = 9 \text{ cm}^2$

Optimal Area:

$A = (9)(4) \div 2 \text{ cm}^2$

$A = 18 \text{ cm}^2$

(b) Current Area:

$A = (7)(3) \div 2 \text{ cm}^2$

$A = 10.5 \text{ cm}^2$

Optimal Area:

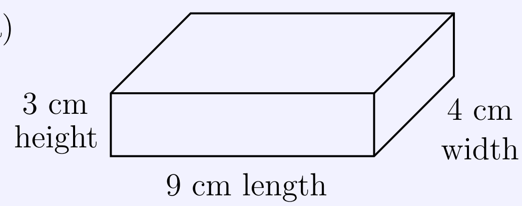
$A = (7)(8) \div 2 \text{ cm}^2$

$A = 28 \text{ cm}^2$

Exercise 5

Calculate the volume of each rectangular prism below.

a)



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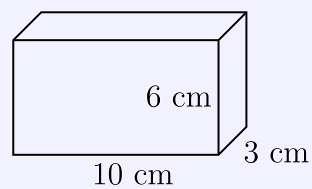


Exercise 5 Solution

(a) $V = (9)(4)(3) \text{ cm}^3$

$V = 108 \text{ cm}^3$

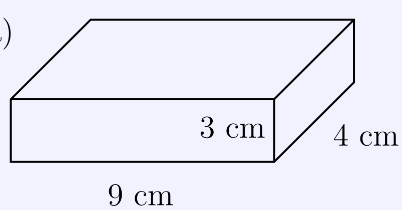
(b) $V = (10)(3)(6) \text{ cm}^3$

$V = 180 \text{ cm}^3$

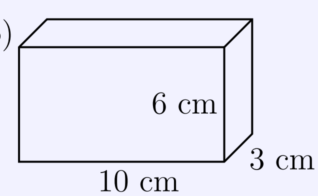
Exercise 6

Calculate the surface area of the following rectangular prisms.

(a)



(b)



Exercise 6 Solution

(a) $SA = 2[(9)(4) + (9)(3) + (4)(3)]\text{ cm}^2$

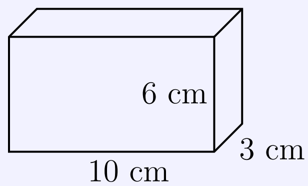
$SA = 150\text{ cm}^2$

(b) $SA = 2[(10)(3) + (10)(6) + (3)(6)]\text{ cm}^2$

$SA = 216\text{ cm}^2$

Exercise 7

Maximize the volume of this rectangular prism while keeping the surface area constant.



Exercise 7 Solution

This solution takes the same form as the Example 4.

$SA = 216\text{ cm}^2$ (from a previous exercise)

We need six equal faces to have an area of 216 cm^2. $\frac{216}{6} = 36$ so each face has an area of 36 cm^2.

Remember that the area of a square is its side length multiplied by itself (since the sides are equal). So we ask, "what number multiplied by itself is equal to 36?"

$6 \times 6 = 36$, so each edge of the cube is 6 cm and $V = 6 \times 6 \times 6 = 216\text{ cm}^3$.

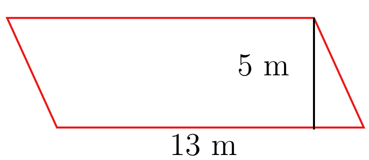
NOTE: It is just a coincidence that the surface area is equal to the maximum volume.

Problem Set Solutions

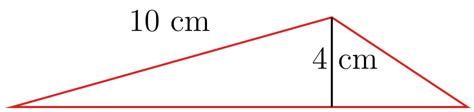
1. Calculate the area of the following shapes.

   (a) A rectangle with sides of length 8 cm and 6 cm.

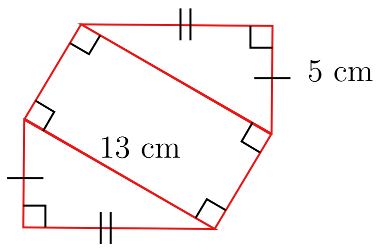
   (b)



(c)



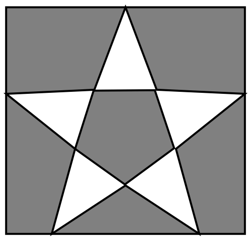
(d) The perimeter of the following shape is 40 cm.



(e) Calculate the area of the grey region given the following information:

The area of the square is 100 cm^2.

The base and height of each triangle is 3 cm and 4 cm, respectively.



Solution:

(a) $A = (8)(6)$

$A = 48 \text{ cm}^2$

(b) $A = (13)(5)$

$A = 65 \text{ m}^2$

(c) $A = \frac{(15)(4)}{2}$

$A = 30 \text{ cm}^2$

(d) Area of One Triangle:

$A = \frac{(12)(5)}{2}$

$A = 30 \text{ cm}^2$

Area of the Rectangle:

The missing side length of the rectangle is 3 cm because all of the other sides add to 34 cm and the total perimeter is 40 cm. Thus, the remaining 6 cm is divided evenly between two sides.

$A = (13)(3)$

$A = 39 \text{ cm}^2$

Total Area: The total area is twice the area of the triangle plus the area of the rectangle. The total area is

$A = 2(30) + 39$

$A = 99 \text{ cm}^2$

(e) To find the grey region, use the area of the entire square and subtract the area of the white triangles.

The area of one white triangle is 6 cm^2 and there are 5 congruent white triangles, so the total white area is 30 cm^2.

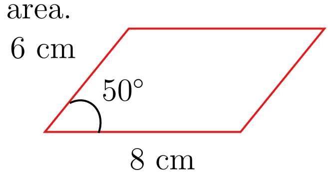
The area of the square minus the area of the white triangles is $100 - 30 = 70$. Therefore, the area of the grey region is 70 cm^2.

2. Determine both the maximum area and the dimensions or angle that produce the maximum area of the following shapes.

(a) A rectangle with a perimeter of 28 cm.

(b) A triangle with a base of 9 cm, a height of 5 cm, and another side of length 6 cm.

(c) The following parallelogram has its side lengths fixed. Change the angle to optimize the area.



Solution:

(a) The maximum area of a rectangle with a fixed perimeter will occur when the rectangle is a square. The total perimeter is 28 cm and there are four equal sides so $\frac{28}{4} = 7$ cm is the length of each side. Therefore, the area is $(7)(7) = 49 \text{ cm}^2$.

(b) The maximum area of a triangle with two given side lengths occurs when the triangle is a right-angled triangle. The right angle forces the second side length to be equal to the height of the triangle. Thus, the base of the optimal triangle is 9 cm and the height is 6 cm.

$A = \frac{(9)(6)}{2}$

$A = 27 \text{ cm}^2$

(c) Since the area of a parallelogram is base times height, we want to maximize these numbers. The base is 8 cm and we cannot change that length by changing the angle. We can change the height by changing the angle. Just like the triangle, we need the angle to be a right angle to maximize the height. This makes the base 8 cm and the height 6 cm.

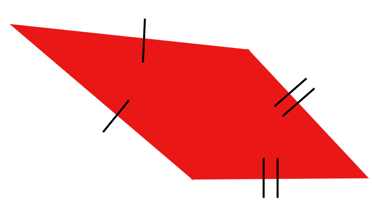
$A = (8)(6)$

$A = 40 \text{ cm}^2$

3. You formed parallelograms and rectangles from pairs of congruent triangles in today's lesson. You might have noticed you can form rectangles from right triangles, but not from acute or obtuse triangles. Why is this?

Solution: You cannot make rectangles from two congruent acute or obtuse triangles because these triangles don't have right angles, so you cannot position the triangles in a way that forms a right angle. Of course, rectangles have four right angles, so it is impossible for the combination of the triangles to be a rectangle.

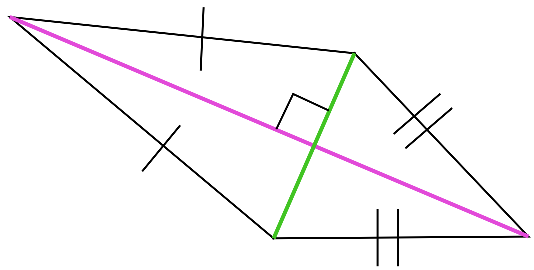
4. (a) Draw two congruent scalene triangles. Record the base and height of your triangles using a ruler. Recall that a scalene triangle is one with no equal sides. Form a **kite** using these two triangles. A kite is a quadrilateral with two pairs of equal adjacent sides.



Example kite

(b) Calculate the area of your kite using the base and height of the triangles you measured.

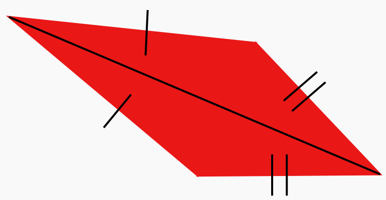
(c) Calculate the area of this kite by dividing it into triangles.



The green line has a length of 8 cm and the purple line has a length of 14 cm.

Solution:

(a)



The example kite from the question has been divided to show the two congruent scalene triangles.

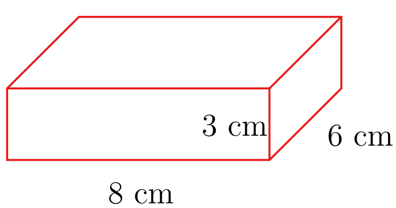
(b) Answers will vary. Use the area of a triangle formula.

(c) This kite can also be seen as two congruent scalene triangles. The base is the purple line and the height is half of the green line.

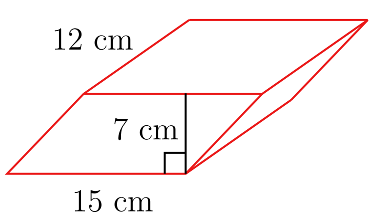
The area of one triangle is $\frac{(14)(4)}{2} = 28 \text{ cm}^2$. Therefore, the area of the full kite is twice this. Thus, the area is 56 cm^2.

5. Calculate the volume of the following solids.

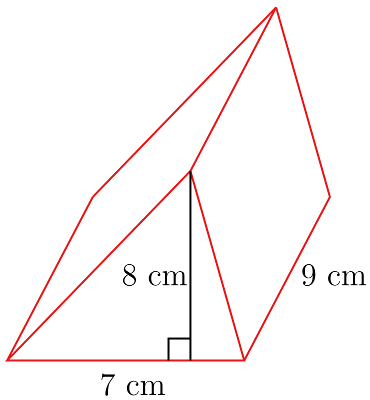
(a)



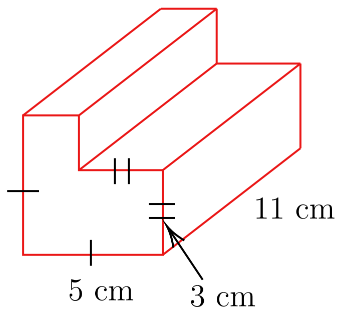
(b)



(c)



(d)



Solution:

(a) $V = (8)(6)(3)$

$V = 144 \text{ cm}^3$

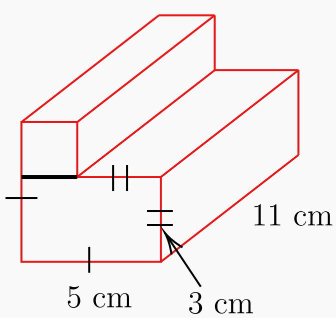
(b) $V = (15)(7)(12)$

$V = 1260 \text{ cm}^3$

(c) $V = \frac{(7)(8)}{2} \times 9$

$V = 252 \text{ cm}^3$

(d)



Divide the base into two rectangles as done above.

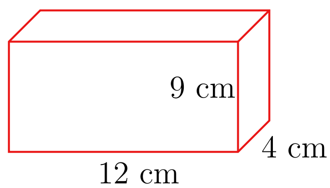
The area of the larger rectangle is $(5)(3) = 15$ cm^2. The sides of the smaller rectangle are 2 cm each, so the area of the small rectangle is 4 cm^2.

This means the total area of the base is 19 cm^2.

$V = (19)(11)$

$V = 209$ cm^3

6. (a) Calculate the surface area of this rectangular prism.



(b) Determine both the maximum volume and the dimensions that produce the maximum volume of the above rectangular prism while keeping surface area constant.

(c) Recalculate the surface area of the optimized rectangular prism. It should be the same as the answer to part (a).

Solution:

(a) $SA = 2[(12)(4) + (12)(9) + (4)(9)]$

$SA = 384$ cm^2

(b) The maximum volume of a rectangular prism with a fixed surface area occurs when the prism is a cube. A cube has six congruent square faces, so each face must have the same surface area. Therefore, each face has an area of $\frac{384}{6} = 64$ cm^2.

A square has equal sides and the area is the side length times itself. $8 \times 8 = 64$ so the each square face has the dimensions 8 cm by 8 cm. Thus, dimensions of the optimal prism is 8 cm by 8 cm by 8 cm. This means the volume is $8 \times 8 \times 8 = 512$ cm^3.

(c) $SA = (8)(8)(6)$

$SA = 384$ cm^2

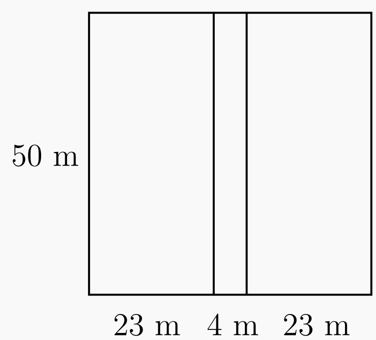
7. Farmer Peter wants to build a fence around two rectangular fields on his property for his crops. He has two types of crop; corn and beans. There must be a 4 meter wide alley between his

corn field and bean field. Both fields must have the same area. He has 200 meters of fencing available and must use all of it. The fencing only surrounds the outside of the fields and not the alley between the fields.

What are the dimensions of each field such that the area of the fields is maximized? What is the area of each field? What is the area of the total fenced area?

Draw a diagram to help with your answer.

Solution:
The length and width of the fencing must be equal because a square optimizes the area of a rectangle with a fixed perimeter. $\frac{200}{4} = 50$ so the dimensions of the fencing is 50 m by 50 m. A 4 meter wide alley will split the fenced area in two equal parts for the two types of crop. Therefore, the width of either field is 23 m. The diagram below models this description.



Therefore, the dimensions of each field are 50 m by 23 m which gives an area of $(50)(23) = 1150$ m^2. The area of the fenced area is $(50)(50) = 2500$ m^2.

8. CHALLENGE PROBLEM

   (a) Brian wants to install an in-ground pool in his backyard. He wants it to be 2 m deep, 10 m long, and as wide as possible given that he has only 110 m^2 of material to make the walls and bottom of the pool. How many meters wide can he make the pool? What is the volume of the pool?

   (b) Brian remembers that he also needs to add a tiled border around the edge of the pool. He wants it to be 1 m wide around the entire pool. He decides to buy 26 m^2 of extra material for the border. Given the pool must be the same depth and same length as part (a), show that the maximum width of the pool with the border is 4.5 m. What is the volume of the pool?

Solution:

(a) We need to make the pool as wide as possible given that Brian has 100 m^2 of surface area materials. The surface area is the sum of the four walls and the bottom of the pool. Let w be the width of the pool.

The bottom of the pool has an area of $10w$. Two walls have an area of $(10)(2) = 20$ each. Two wall have an area of $2w$ each.

$SA = 10w + 2(10)(2) + 2(2w)$

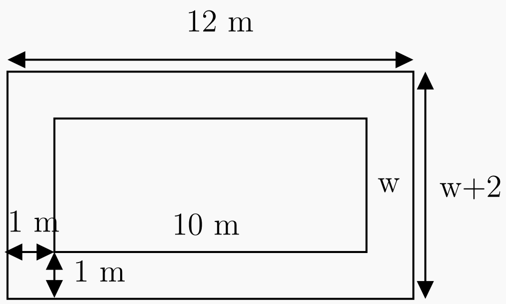
$110 = 10w + 40 + 4w$

$70 = 14w$

$w = 5m$

Thus, the widest the pool can be is 5 m.

(b) The following diagram gives a bird's-eye view of the pool and tiled border.



The base of the pool plus the tiled border has an area of $12(w+2) = 12w+24$. Two walls still have an area of $(10)(2) = 20$ each. The other two walls still have an area of $2w$ each.

$SA = 12w + 24 + 2(20) + 2(2w)$

$110 + 26 = 12w + 24 + 40 + 4w$

$136 = 16w + 64$

$72 = 16w$

$w = 4.5$ m

Thus, the width of the pool is 4.5 m and the volume is $(10)(4.5)(2) = 90$ m^3.